Supplementary Material for

"The ecology of asexual pairwise interactions: A generalized law of mass action"

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Functions ϕ's for the Holling-type-II M-prey-one-predator model described in Box. 3

The prey g-function for M=1,2,3,4 (see Eq. (B3.1))

```
g1[n_, N_, x_, xp_] := r[xp] - c[xp, x] n - a[xp] N (1 + a[x] h[x] n)^-1;
g2[n1_, n2_, N_, x1_, x2_, xp_] :=
  r[xp] - c[xp, x1] n1 - c[xp, x2] n2 - a[xp] N (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2)^-1;
g3[n1_, n2_, n3_, N_, x1_, x2_, x3_, xp_] := r[xp] - c[xp, x1] n1 - c[xp, x2] n2 -
   c[xp, x3] n3 - a[xp] N (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2 + a[x3] h[x3] n3)^-1;
g4[n1_, n2_, n3_, n4_, N_, x1_, x2_, x3_, x4_, xp_] :=
  r[xp] - c[xp, x1] n1 - c[xp, x2] n2 - c[xp, x3] n3 - c[xp, x4] n4 -
   a[xp] N (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2 + a[x3] h[x3] n3 + a[x4] h[x4] n4)^-1;
```

Functions ϕ's appearing in first derivatives

Function $\phi_{1,1}$

Recall the rule defined at the end of Sect. 2.3 to uniquely identify the functions ϕ's. For $\phi_{1,1}$, take the first derivative of g_2 w.r.t. x_1 ($M = k + 1$ where k is the number of perturbed resident strategies) and evaluate the resulting expression at $x_1 = x_2 = x$

```
tmp1 = Simplify[D[g2[n1, n2, N, x1, x2, xp], {x1, 1}, {x2, 0}] /. {x1 → x, x2 → x}]
```

$$\frac{N\, n1\, a[xp]\, (h[x]\, a'[x] + a[x]\, h'[x])}{(1 + (n1 + n2)\, a[x]\, h[x])^2} - n1\, c^{(0,1)}[xp, x]$$

Note that n_2 only appear in sum with n_1, because no differentiation is taken w.r.t. x_2. Then, substituting n_2 with $n - n_1$ and simplification introduce the symbol n wherever required by property P4

```
tmp2 = Simplify[Dx1g2 /. {n2 → n - n1}]
```

```
Dx1g2
```

Function $\phi_{1,1}$ is then the resulting coefficient of n_1

```
ϕ1,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp2, n1]]
```

```
0
```

Functions ϕ's appearing in second derivatives

Functions $\phi_{2,1}$ and $\phi_{2,2}$ appearing in second pure derivatives

For $\phi_{2,1}$ and $\phi_{2,2}$, take the x_1-second-derivative of g_2 evaluated at $x_1 = x_2 = x$ and substitute n_2 with $n - n_1$

```
tmp = Simplify[
   D[g2[n1, n2, N, x1, x2, xp], {x1, 2}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

Functions $\phi_{2,1}$ and $\phi_{2,2}$ are then the resulting coefficients of n_1 and n_1^2

```
ϕ2,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1]]
```

$$\frac{N\,a[xp]\,(2\,a'[x]\,h'[x]+h[x]\,a''[x]+a[x]\,h''[x])-(1+n\,a[x]\,h[x])^2\,c^{(0,2)}[xp,x]}{(1+n\,a[x]\,h[x])^2}$$

```
ϕ2,2[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1^2]]
```

$$-\frac{2\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^2}{(1+n\,a[x]\,h[x])^3}$$

Function $\phi_{1,1,1,1}$ appearing in second mixed derivatives

For $\phi_{1,1,1,1}$, take the (x_1, x_2)-second-mixed-derivative of g_3 evaluated at $x_1 = x_2 = x_3 = x$ and substitute n_3 with $n - n_1 - n_2$

```
tmp = Simplify[D[g3[n1, n2, n3, N, x1, x2, x3, xp], {x1, 1}, {x2, 1}] /.
    {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}]
```

$$-\frac{2\,N\,n1\,n2\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^2}{(1+n\,a[x]\,h[x])^3}$$

Function $\phi_{1,1,1,1}$ is then the resulting coefficient of the monomial $n_1\,n_2$

```
ϕ1,1,1,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$-\frac{2\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^2}{(1+n\,a[x]\,h[x])^3}$$

Functions ϕ's appearing in third derivatives

Functions $\phi_{3,1}$, $\phi_{3,2}$, and $\phi_{3,3}$ appearing in third pure derivatives

```
tmp = Simplify[
   D[g2[n1, n2, N, x1, x2, xp], {x1, 3}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ϕ3,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1]]
```

$$\frac{1}{(1+n\,a[x]\,h[x])^2}\left(N\,a[xp]\left(3\,h'[x]\,a''[x]+3\,a'[x]\,h''[x]+h[x]\,a^{(3)}[x]+a[x]\,h^{(3)}[x]\right)-(1+n\,a[x]\,h[x])^2\,c^{(0,3)}[xp,x]\right)$$

```
ϕ3,2[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1^2]]
```

$$-\frac{6\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])\,(2\,a'[x]\,h'[x]+h[x]\,a''[x]+a[x]\,h''[x])}{(1+n\,a[x]\,h[x])^3}$$

```
φ3,3[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1^3]]
```

$$\frac{6\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^3}{(1+n\,a[x]\,h[x])^4}$$

Functions $\phi_{2,1,1,1}$ and $\phi_{2,1,2,1}$ appearing in third mixed derivatives w.r.t. two strategies

```
tmp = Simplify[D[g3[n1, n2, n3, N, x1, x2, x3, xp], {x1, 2}, {x2, 1}] /.
    {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
```

```
φ2,1,1,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$-\frac{2\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])\,(2\,a'[x]\,h'[x]+h[x]\,a''[x]+a[x]\,h''[x])}{(1+n\,a[x]\,h[x])^3}$$

```
φ2,1,2,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1^2 n2]]
```

$$\frac{6\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^3}{(1+n\,a[x]\,h[x])^4}$$

Function $\phi_{1,1,1,1,1,1}$ appearing in third mixed derivatives w.r.t. three strategies

```
tmp =
  Simplify[D[g4[n1, n2, n3, n4, N, x1, x2, x3, x4, xp], {x1, 1}, {x2, 1}, {x3, 1}] /.
    {x1 → x, x2 → x, x3 → x, x4 → x} /. {n4 → n - n1 - n2 - n3}];
```

```
φ1,1,1,1,1,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2 n3]]
```

$$\frac{6\,N\,a[xp]\,(h[x]\,a'[x]+a[x]\,h'[x])^3}{(1+n\,a[x]\,h[x])^4}$$

Symmetries due to property P3

Let's now check the symmetries among 2-index ϕ's.

$\phi_{1,2,1,1} = \phi_{2,1,1,1}$

First, function $\phi_{1,2,1,1}$ must be defined, analogously to what done above for $\phi_{2,1,1,1}$

```
tmp = Simplify[D[g3[n1, n2, n3, N, x1, x2, x3, xp], {x1, 1}, {x2, 2}] /.
    {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
φ1,2,1,1[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]];
```

Then the equivalence between $\phi_{1,2,1,1}$ and $\phi_{2,1,1,1}$ can be checked by simplifying the difference

```
Simplify[φ1,2,1,1[n, N, x, xp] - φ2,1,1,1[n, N, x, xp]]
```

0

The other symmetry is similarly checked below.

$\phi_{1,2,1,2} = \phi_{2,1,2,1}$

```
φ1,2,1,2[n_, N_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2^2]];
Simplify[φ1,2,1,2[n, N, x, xp] - φ2,1,2,1[n, N, x, xp]]
```

0

Links between k- and k'-index-ϕ's, k'<k (see Eqs. (10a-d))

$\phi_{1,1,1,1} = \phi_{2,2}$

```
Simplify[ϕ1,1,1,1[n, N, x, xp] - ϕ2,2[n, N, x, xp]]
```

0

$\phi_{2,1,1,1} = \phi_{3,2} / 3$

```
Simplify[ϕ2,1,1,1[n, N, x, xp] - ϕ3,2[n, N, x, xp] / 3]
```

0

$\phi_{2,1,2,1} = \phi_{3,3}$

```
Simplify[ϕ2,1,2,1[n, N, x, xp] - ϕ3,3[n, N, x, xp]]
```

0

$\phi_{1,1,1,1,1,1} = \phi_{3,3}$

```
Simplify[ϕ1,1,1,1,1,1[n, N, x, xp] - ϕ3,3[n, N, x, xp]]
```

0

Consistency with the monomorphic g-function

To verify that the functions ϕ's are indeed M-independent, property P4 is checked below (up to order 3) on the monomorphic g-function

```
Simplify[D[g1[n, N, x, xp], {x, 1}] - ϕ1,1[n, N, x, xp] n]
```

$$\frac{n\,N\,a[xp]\,(h[x]\,a'[x] + a[x]\,h'[x])}{(1 + n\,a[x]\,h[x])^2} - n\,c^{(0,1)}[xp, x]$$

```
Simplify[D[g1[n, N, x, xp], {x, 2}] - (ϕ2,1[n, N, x, xp] n + ϕ2,2[n, N, x, xp] n^2)]
```

0

```
Simplify[D[g1[n, N, x, xp], {x, 3}] -
  (ϕ3,1[n, N, x, xp] n + ϕ3,2[n, N, x, xp] n^2 + ϕ3,3[n, N, x, xp] n^3)]
```

0

Other examples of property P4

```
Simplify[
 (D[g2[n1, n2, N, x1, x2, xp], {x2, 1}] /. {x1 → x, x2 → x}) - ϕ1,1[n1 + n2, N, x, xp] n2]
```

$$\frac{N\,n2\,a[xp]\,(h[x]\,a'[x] + a[x]\,h'[x])}{(1 + (n1 + n2)\,a[x]\,h[x])^2} - n2\,c^{(0,1)}[xp, x]$$

```
Simplify[(D[g2[n1, n2, N, x1, x2, xp], {x1, 1}, {x2, 1}] /. {x1 → x, x2 → x}) -
  ϕ1,1,1,1[n1 + n2, N, x, xp] n1 n2]
```

0

```
Simplify[(D[g2[n1, n2, N, x1, x2, xp], {x1, 1}, {x2, 2}] /. {x1 → x, x2 → x}) -
  (ϕ1,2,1,1[n1 + n2, N, x, xp] n1 n2 + ϕ1,2,1,2[n1 + n2, N, x, xp] n1 n2^2)]
```

0

```
Simplify[(D[g3[n1, n2, n3, N, x1, x2, x3, xp], {x2, 1}] /. {x1 → x, x2 → x, x3 → x}) -
  ϕ1,1[n1 + n2 + n3, N, x, xp] n2]
```

$$\frac{\mathrm{N}\,\mathrm{n2}\,\mathrm{a}[\mathrm{xp}]\,(\mathrm{h}[\mathrm{x}]\,\mathrm{a}'[\mathrm{x}] + \mathrm{a}[\mathrm{x}]\,\mathrm{h}'[\mathrm{x}])}{(1 + (\mathrm{n1} + \mathrm{n2} + \mathrm{n3})\,\mathrm{a}[\mathrm{x}]\,\mathrm{h}[\mathrm{x}])^2} - \mathrm{n2}\,\mathrm{c}^{(0,1)}[\mathrm{xp}, \mathrm{x}]$$

```
Simplify[(D[g4[n1, n2, n3, n4, N, x1, x2, x3, x4, xp], {x2, 1}, {x3, 1}] /.
    {x1 → x, x2 → x, x3 → x, x4 → x}) - ϕ1,1,1,1[n1 + n2 + n3 + n4, N, x, xp] n2 n3]
```

0

Clear symbols

```
Clear[g1, g2, g3, g4, tmp]
Unset[{ϕ1,1[n_, N_, x_, xp_], ϕ2,1[n_, N_, x_, xp_], ϕ2,2[n_, N_, x_, xp_],
   ϕ1,1,1,1[n_, N_, x_, xp_], ϕ3,1[n_, N_, x_, xp_], ϕ3,2[n_, N_, x_, xp_],
   ϕ3,3[n_, N_, x_, xp_], ϕ2,1,1,1[n_, N_, x_, xp_], ϕ2,1,2,1[n_, N_, x_, xp_],
   ϕ1,1,1,1,1,1[n_, N_, x_, xp_], ϕ1,2,1,1[n_, N_, x_, xp_], ϕ1,2,1,2[n_, N_, x_, xp_]}];
```

Functions ψ's for the Holling-type-II M-prey-one-predator model described in Boxes. 7 and 8

The predator per-capita growth rate for M=1,2,3,4 (see Eq. (B7.1))

```
F1[n_, x_] := e[x] a[x] n (1 + a[x] h[x] n)^-1 - d;
F2[n1_, n2_, x1_, x2_] :=
  (e[x1] a[x1] n1 + e[x2] a[x2] n2) (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2)^-1 - d;
F3[n1_, n2_, n3_, x1_, x2_, x3_] := (e[x1] a[x1] n1 + e[x2] a[x2] n2 + e[x3] a[x3] n3)
    (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2 + a[x3] h[x3] n3)^-1 - d;
F4[n1_, n2_, n3_, n4_, x1_, x2_, x3_, x4_] :=
  (e[x1] a[x1] n1 + e[x2] a[x2] n2 + e[x3] a[x3] n3 + e[x4] a[x4] n4)
    (1 + a[x1] h[x1] n1 + a[x2] h[x2] n2 + a[x3] h[x3] n3 + a[x4] h[x4] n4)^-1 - d;
```

Functions ψ's appearing in first derivatives

Function $\psi_{1,1}$

```
tmp = Simplify[
   D[F2[n1, n2, x1, x2], {x1, 1}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
ψ1,1[n_, x_] = Simplify[Coefficient[tmp, n1]]
```

$$\frac{\mathrm{a}[\mathrm{x}]\,(1 + \mathrm{n}\,\mathrm{a}[\mathrm{x}]\,\mathrm{h}[\mathrm{x}])\,\mathrm{e}'[\mathrm{x}] + \mathrm{e}[\mathrm{x}]\,(\mathrm{a}'[\mathrm{x}] - \mathrm{n}\,\mathrm{a}[\mathrm{x}]^2\,\mathrm{h}'[\mathrm{x}])}{(1 + \mathrm{n}\,\mathrm{a}[\mathrm{x}]\,\mathrm{h}[\mathrm{x}])^2}$$

Functions ψ's appearing in second derivatives

Functions $\psi_{2,1}$ and $\psi_{2,2}$ appearing in second pure derivatives

```
tmp = Simplify[
   D[F2[n1, n2, x1, x2], {x1, 2}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ψ2,1[n_, x_] = Simplify[Coefficient[tmp, n1]]
```

$$\frac{1}{(1+n\,a[x]\,h[x])^2}\left(2\,a'[x]\,((1+n\,a[x]\,h[x])\,e'[x]-n\,a[x]\,e[x]\,h'[x])+a[x]\,(1+n\,a[x]\,h[x])\,e''[x]+e[x]\,\left(a''[x]-n\,a[x]^2\,h''[x]\right)\right)$$

```
ψ2,2[n_, x_] = Simplify[Coefficient[tmp, n1^2]]
```

$$-\frac{2\,(h[x]\,a'[x]+a[x]\,h'[x])\,(a[x]\,(1+n\,a[x]\,h[x])\,e'[x]+e[x]\,(a'[x]-n\,a[x]^2\,h'[x]))}{(1+n\,a[x]\,h[x])^3}$$

Function $\psi_{1,1,1,1}$ appearing in second mixed derivatives

```
tmp = Simplify[D[F3[n1, n2, n3, x1, x2, x3], {x1, 1}, {x2, 1}] /.
     {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
```

```
ψ1,1,1,1[n_, x_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$-\frac{2\,(h[x]\,a'[x]+a[x]\,h'[x])\,(a[x]\,(1+n\,a[x]\,h[x])\,e'[x]+e[x]\,(a'[x]-n\,a[x]^2\,h'[x]))}{(1+n\,a[x]\,h[x])^3}$$

Functions ψ's appearing in third derivatives

Functions $\psi_{3,1}$, $\psi_{3,2}$, and $\psi_{3,3}$ appearing in third pure derivatives

```
tmp = Simplify[
   D[F2[n1, n2, x1, x2], {x1, 3}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ψ3,1[n_, x_] = Simplify[Coefficient[tmp, n1]]
```

$$\frac{1}{(1+n\,a[x]\,h[x])^2}\left(3\,(1+n\,a[x]\,h[x])\,e'[x]\,a''[x]+3\,a'[x]\,e''[x]+e[x]\,a^{(3)}[x]+a[x]\,\left(3\,n\,h[x]\,a'[x]\,e''[x]-3\,n\,e[x]\,(h'[x]\,a''[x]+a'[x]\,h''[x])+e^{(3)}[x]\right)+n\,a[x]^2\,\left(h[x]\,e^{(3)}[x]-e[x]\,h^{(3)}[x]\right)\right)$$

```
ψ3,2[n_, x_] = Simplify[Coefficient[tmp, n1^2]]
```

$$-\frac{1}{(1+n\,a[x]\,h[x])^3}3\,\Big(n\,a[x]\,h[x]^2\,\left(2\,a'[x]^2\,e'[x]+a[x]\,e'[x]\,a''[x]+a[x]\,a'[x]\,e''[x]\right)+e[x]\,\Big(2\,a'[x]^2\,h'[x]+a[x]\,a'[x]\,\left(-4\,n\,a[x]\,h'[x]^2+h''[x]\right)+a[x]\,h'[x]\,\left(a''[x]-2\,n\,a[x]^2\,h''[x]\right)\Big)+a[x]\,(4\,a'[x]\,e'[x]\,h'[x]+a[x]\,(h'[x]\,e''[x]+e'[x]\,h''[x]))+h[x]\,\Big(2\,a'[x]^2\,(e'[x]-n\,a[x]\,e[x]\,h'[x])+a[x]\,\left(n\,a[x]\,h'[x]\,(-e[x]\,a''[x]+a[x]\,e''[x])+e'[x]\,\left(a''[x]+n\,a[x]^2\,h''[x]\right)\right)+a'[x]\,\left(2\,e[x]\,a''[x]+a[x]\,e''[x]+n\,a[x]^2\,(4\,e'[x]\,h'[x]-e[x]\,h''[x])\right)\Big)\Big)$$

```
ψ3,3[n_, x_] = Simplify[Coefficient[tmp, n1^3]]
```

$$\frac{6\,(h[x]\,a'[x]+a[x]\,h'[x])^2\,(a[x]\,(1+n\,a[x]\,h[x])\,e'[x]+e[x]\,(a'[x]-n\,a[x]^2\,h'[x]))}{(1+n\,a[x]\,h[x])^4}$$

Functions $\psi_{2,1,1,1}$ and $\psi_{2,1,2,1}$ appearing in third mixed derivatives w.r.t. two strategies

```
tmp = Simplify[D[F3[n1, n2, n3, x1, x2, x3], {x1, 2}, {x2, 1}] /.
      {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
```

```
ψ2,1,1,1[n_, x_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$-\frac{1}{(1+n\,a[x]\,h[x])^3}\Big(n\,a[x]\,h[x]^2\,\big(2\,a'[x]^2\,e'[x]+a[x]\,e'[x]\,a''[x]+a[x]\,a'[x]\,e''[x]\big)+$$
$$e[x]\,\big(2\,a'[x]^2\,h'[x]+a[x]\,a'[x]\,\big(-4\,n\,a[x]\,h'[x]^2+h''[x]\big)+$$
$$a[x]\,h'[x]\,\big(a''[x]-2\,n\,a[x]^2\,h''[x]\big)\big)+$$
$$a[x]\,(4\,a'[x]\,e'[x]\,h'[x]+a[x]\,(h'[x]\,e''[x]+e'[x]\,h''[x]))+$$
$$h[x]\,\big(2\,a'[x]^2\,(e'[x]-n\,a[x]\,e[x]\,h'[x])+$$
$$a[x]\,\big(n\,a[x]\,h'[x]\,(-e[x]\,a''[x]+a[x]\,e''[x])+e'[x]\,\big(a''[x]+n\,a[x]^2\,h''[x]\big)\big)+$$
$$a'[x]\,\big(2\,e[x]\,a''[x]+a[x]\,e''[x]+n\,a[x]^2\,(4\,e'[x]\,h'[x]-e[x]\,h''[x])\big)\big)\Big)$$

```
ψ2,1,2,1[n_, x_] = Simplify[Coefficient[tmp, n1^2 n2]]
```

$$\frac{6\,(h[x]\,a'[x]+a[x]\,h'[x])^2\,(a[x]\,(1+n\,a[x]\,h[x])\,e'[x]+e[x]\,(a'[x]-n\,a[x]^2\,h'[x]))}{(1+n\,a[x]\,h[x])^4}$$

Function $\psi_{1,1,1,1,1,1}$ appearing in third mixed derivatives w.r.t. three strategies

```
tmp = Simplify[D[F4[n1, n2, n3, n4, x1, x2, x3, x4], {x1, 1}, {x2, 1}, {x3, 1}] /.
      {x1 → x, x2 → x, x3 → x, x4 → x} /. {n4 → n - n1 - n2 - n3}];
```

```
ψ1,1,1,1,1,1[n_, x_] = Simplify[Coefficient[tmp, n1 n2 n3]]
```

$$\frac{6\,(h[x]\,a'[x]+a[x]\,h'[x])^2\,(a[x]\,(1+n\,a[x]\,h[x])\,e'[x]+e[x]\,(a'[x]-n\,a[x]^2\,h'[x]))}{(1+n\,a[x]\,h[x])^4}$$

Symmetries due to property P3

$\psi_{1,2,1,1} = \psi_{2,1,1,1}$

```
tmp = Simplify[D[F3[n1, n2, n3, x1, x2, x3], {x1, 1}, {x2, 2}] /.
      {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
ψ1,2,1,1[n_, x_] = Simplify[Coefficient[tmp, n1 n2]];
```

```
Simplify[ψ1,2,1,1[n, x] - ψ2,1,1,1[n, x]]
```

0

$\psi_{1,2,1,2} = \psi_{2,1,2,1}$

```
ψ1,2,1,2[n_, x_] = Simplify[Coefficient[tmp, n1 n2^2]];
Simplify[ψ1,2,1,2[n, x] - ψ2,1,2,1[n, x]]
```

0

Links between k- and k'-index-ψ's, k'<k (see Eqs. (A8a-d))

$\psi_{1,1,1,1} = \psi_{2,2}$

```
Simplify[ψ_{1,1,1,1}[n, x] - ψ_{2,2}[n, x]]
```

0

$\psi_{2,1,1,1} = \psi_{3,2} / 3$

```
Simplify[ψ_{2,1,1,1}[n, x] - ψ_{3,2}[n, x] / 3]
```

0

$\psi_{2,1,2,1} = \psi_{3,3}$

```
Simplify[ψ_{2,1,2,1}[n, x] - ψ_{3,3}[n, x]]
```

0

$\psi_{1,1,1,1,1,1} = \psi_{3,3}$

```
Simplify[ψ_{1,1,1,1,1,1}[n, x] - ψ_{3,3}[n, x]]
```

0

Consistency with the monomorphic F-function

```
Simplify[D[F1[n, x], {x, 1}] - ψ_{1,1}[n, x] n]
```

0

```
Simplify[D[F1[n, x], {x, 2}] - (ψ_{2,1}[n, x] n + ψ_{2,2}[n, x] n^2)]
```

0

```
Simplify[D[F1[n, x], {x, 3}] - (ψ_{3,1}[n, x] n + ψ_{3,2}[n, x] n^2 + ψ_{3,3}[n, x] n^3)]
```

0

Other examples of property P4

```
Simplify[(D[F2[n1, n2, x1, x2], {x2, 1}] /. {x1 → x, x2 → x}) - ψ_{1,1}[n1 + n2, x] n2]
```

0

```
Simplify[(D[F2[n1, n2, x1, x2], {x1, 1}, {x2, 1}] /. {x1 → x, x2 → x}) -
  ψ_{1,1,1,1}[n1 + n2, x] n1 n2]
```

0

```
Simplify[(D[F2[n1, n2, x1, x2], {x1, 1}, {x2, 2}] /. {x1 → x, x2 → x}) -
  (ψ_{1,2,1,1}[n1 + n2, x] n1 n2 + ψ_{1,2,1,2}[n1 + n2, x] n1 n2^2)]
```

0

```
Simplify[(D[F3[n1, n2, n3, x1, x2, x3], {x2, 1}] /. {x1 → x, x2 → x, x3 → x}) -
  ψ_{1,1}[n1 + n2 + n3, x] n2]
```

0

```
Simplify[(D[F4[n1, n2, n3, n4, x1, x2, x3, x4], {x2, 1}, {x3, 1}] /.
    {x1 → x, x2 → x, x3 → x, x4 → x}) - ψ_{1,1,1,1}[n1 + n2 + n3 + n4, x] n2 n3]
```

0

Clear symbols

```
Clear[F1, F2, F3, F4, tmp]
Unset[{ψ1,1[n_, x_], ψ2,1[n_, x_], ψ2,2[n_, x_], ψ1,1,1,1[n_, x_],
   ψ3,1[n_, x_], ψ3,2[n_, x_], ψ3,3[n_, x_], ψ2,1,1,1[n_, x_],
   ψ2,1,2,1[n_, x_], ψ1,1,1,1,1,1[n_, x_], ψ1,2,1,1[n_, x_], ψ1,2,1,2[n_, x_]}];
```

Functions ϕ's for the model of cannibalism described in Box. 5

The species g-function for M=1,2,3,4 (see Eq. (B5.1))

```
g1[n1_, x1_, xp_] :=
  e (a0[xp] n0 + a[xp, x1] n1) / (1 + a0[xp] h[xp] n0 + a[xp, x1] h[xp] n1) -
   a[x1, xp] n1 / (1 + a0[x1] h[x1] n0 + a[x1, x1] h[x1] n1) - c (n1 + n2);
g2[n1_, n2_, x1_, x2_, xp_] := e (a0[xp] n0 + a[xp, x1] n1 + a[xp, x2] n2) /
     (1 + a0[xp] h[xp] n0 + a[xp, x1] h[xp] n1 + a[xp, x2] h[xp] n2) -
   a[x1, xp] n1 / (1 + a0[x1] h[x1] n0 + a[x1, x1] h[x1] n1 + a[x1, x2] h[x1] n2) -
   a[x2, xp] n2 / (1 + a0[x2] h[x2] n0 + a[x2, x1] h[x2] n1 + a[x2, x2] h[x2] n2) -
   c (n1 + n2);
g3[n1_, n2_, n3_, x1_, x2_, x3_, xp_] :=
  e (a0[xp] n0 + a[xp, x1] n1 + a[xp, x2] n2 + a[xp, x3] n3) / (1 + a0[xp] h[xp] n0 +
       a[xp, x1] h[xp] n1 + a[xp, x2] h[xp] n2 + a[xp, x3] h[xp] n3) -
   a[x1, xp] n1 / (1 + a0[x1] h[x1] n0 + a[x1, x1] h[x1] n1 + a[x1, x2] h[x1] n2 +
       a[x1, x3] h[x1] n3) - a[x2, xp] n2 / (1 + a0[x2] h[x2] n0 +
       a[x2, x1] h[x2] n1 + a[x2, x2] h[x2] n2 + a[x2, x3] h[x2] n3) -
   a[x3, xp] n3 / (1 + a0[x3] h[x3] n0 + a[x3, x1] h[x3] n1 +
       a[x3, x2] h[x3] n2 + a[x3, x3] h[x3] n3) - c (n1 + n2 + n3);
g4[n1_, n2_, n3_, n4_, x1_, x2_, x3_, x4_, xp_] :=
  e (a0[xp] n0 + a[xp, x1] n1 + a[xp, x2] n2 + a[xp, x3] n3 + a[xp, x4] n4) /
     (1 + a0[xp] h[xp] n0 + a[xp, x1] h[xp] n1 +
       a[xp, x2] h[xp] n2 + a[xp, x3] h[xp] n3 + a[xp, x4] h[xp] n4) -
   a[x1, xp] n1 / (1 + a0[x1] h[x1] n0 + a[x1, x1] h[x1] n1 + a[x1, x2] h[x1] n2 +
       a[x1, x3] h[x1] n3 + a[x1, x4] h[x1] n4) -
   a[x2, xp] n2 / (1 + a0[x2] h[x2] n0 + a[x2, x1] h[x2] n1 + a[x2, x2] h[x2] n2 +
       a[x2, x3] h[x2] n3 + a[x2, x4] h[x2] n4) -
   a[x3, xp] n3 / (1 + a0[x3] h[x3] n0 + a[x3, x1] h[x3] n1 + a[x3, x2] h[x3] n2 +
       a[x3, x3] h[x3] n3 + a[x3, x4] h[x3] n4) -
   a[x4, xp] n4 / (1 + a0[x4] h[x4] n0 + a[x4, x1] h[x4] n1 + a[x4, x2] h[x4] n2 +
       a[x4, x3] h[x4] n3 + a[x4, x4] h[x4] n4) - c (n1 + n2 + n3 + n4);
```

Functions ϕ's appearing in first derivatives

Function $\phi_{1,1}$

```
tmp = Simplify[
   D[g2[n1, n2, x1, x2, xp], {x1, 1}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ϕ1,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1]]
```

```
(a[x, xp]
    ((n a[x, x] + n0 a0[x]) h′[x] + h[x] (n0 a0′[x] + n (a^(0,1)[x, x] + a^(1,0)[x, x]))) +
   ((1 + n a[x, x] h[x] + n0 a0[x] h[x]) (e (1 + n a[x, x] h[x] + n0 a0[x] h[x])
        a^(0,1)[xp, x] - (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2 a^(1,0)[x, xp])) /
    (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2) / (1 + n a[x, x] h[x] + n0 a0[x] h[x])^2
```

Functions ϕ’s appearing in second derivatives

Functions $\phi_{2,1}$ and $\phi_{2,2}$ appearing in second pure derivatives

```
tmp = Simplify[
   D[g2[n1, n2, x1, x2, xp], {x1, 2}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ϕ2,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1]]
```

```
(a[x, xp] (2 n0 a0′[x] h′[x] - 2 n^2 a[x, x]^2 h′[x]^2 - 4 n n0 a[x, x] a0[x] h′[x]^2 -
        2 n0^2 a0[x]^2 h′[x]^2 + n a[x, x] h″[x] + n0 a0[x] h″[x] + 2 n h′[x] a^(1,0)[x, x] +
        h[x] (n0 a0″[x] + n^2 a[x, x]^2 h″[x] + n0^2 a0[x]^2 h″[x] + n a^(0,2)[x, x] -
           2 n0 a0[x] h′[x] (n0 a0′[x] + n a^(1,0)[x, x]) - 2 n a[x, x]
             (n0 a0′[x] h′[x] - n0 a0[x] h″[x] + n h′[x] a^(1,0)[x, x]) + n a^(2,0)[x, x]) +
        h[x]^2 (-2 n0^2 a0′[x]^2 + n0^2 a0[x] a0″[x] + n n0 a0[x] a^(0,2)[x, x] -
           4 n n0 a0′[x] a^(1,0)[x, x] - 2 n^2 a^(1,0)[x, x]^2 + n n0 a0[x] a^(2,0)[x, x] +
           n a[x, x] (n0 a0″[x] + n (a^(0,2)[x, x] + a^(2,0)[x, x])))) +
     ((1 + n a[x, x] h[x] + n0 a0[x] h[x]) (e (1 + n a[x, x] h[x] + n0 a0[x] h[x])^2
           a^(0,2)[xp, x] + (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2
           (2 n a[x, x] h′[x] a^(1,0)[x, xp] + 2 n0 a0[x] h′[x] a^(1,0)[x, xp] -
             a^(2,0)[x, xp] + h[x] (2 n0 a0′[x] a^(1,0)[x, xp] +
                 2 n a^(1,0)[x, x] a^(1,0)[x, xp] - (n a[x, x] + n0 a0[x]) a^(2,0)[x, xp])))) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2) / (1 + n a[x, x] h[x] +
     n0 a0[x] h[x])^3
```

```
ϕ2,2[n_, x_, xp_] = Simplify[Coefficient[tmp, n1^2]]
```

```
(2 (((1 + n a[x, x] h[x] + n0 a0[x] h[x]) (h[x] a^(0,1)[x, x] a^(1,0)[x, xp] +
                3 (n a[xp, x] + n0 a0[xp])^2 h[x] h[xp]^2 a^(0,1)[x, x] a^(1,0)[x, xp] +
                (n a[xp, x] + n0 a0[xp])^3 h[x] h[xp]^3 a^(0,1)[x, x] a^(1,0)[x, xp] -
                h[xp] (e (1 + n a[x, x] h[x] + n0 a0[x] h[x])^2 a^(0,1)[xp, x]^2 -
                    3 (n a[xp, x] + n0 a0[xp]) h[x] a^(0,1)[x, x] a^(1,0)[x, xp]))) /
           (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^3 + a[x, xp] (h′[x] a^(0,1)[x, x] +
             h[x] (-n a[x, x] h′[x] a^(0,1)[x, x] - n0 a0[x] h′[x] a^(0,1)[x, x] + a^(1,1)[x, x]) +
             h[x]^2 (-2 n0 a0′[x] a^(0,1)[x, x] - n a^(0,1)[x, x]^2 - 2 n a^(0,1)[x, x] a^(1,0)[x, x] +
                 (n a[x, x] + n0 a0[x]) a^(1,1)[x, x])))) / (1 + n a[x, x] h[x] + n0 a0[x] h[x])^3
```

Function $\phi_{1,1,1,1}$ appearing in second mixed derivatives

```
tmp = Simplify[D[g3[n1, n2, n3, x1, x2, x3, xp], {x1, 1}, {x2, 1}] /.
      {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
```

```
ϕ1,1,1,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$
\begin{aligned}
&2\Bigg(\frac{e\, n\, a[xp,x]\, h[xp]^2\, a^{(0,1)}[xp,x]^2}{(1+n\, a[xp,x]\, h[xp]+n0\, a0[xp]\, h[xp])^3}+\\
&\frac{e\, n0\, a0[xp]\, h[xp]^2\, a^{(0,1)}[xp,x]^2}{(1+n\, a[xp,x]\, h[xp]+n0\, a0[xp]\, h[xp])^3}-\frac{e\, h[xp]\, a^{(0,1)}[xp,x]^2}{(1+n\, a[xp,x]\, h[xp]+n0\, a0[xp]\, h[xp])^2}+\\
&\frac{h[x]\, a^{(0,1)}[x,x]\, a^{(1,0)}[x,xp]}{(1+n\, a[x,x]\, h[x]+n0\, a0[x]\, h[x])^2}+\big(a[x,xp]\,\big(h'[x]\, a^{(0,1)}[x,x]+\\
&\quad h[x]\,\big(-n\, a[x,x]\, h'[x]\, a^{(0,1)}[x,x]-n0\, a0[x]\, h'[x]\, a^{(0,1)}[x,x]+a^{(1,1)}[x,x]\big)+\\
&\quad h[x]^2\,\big(-2\, n0\, a0'[x]\, a^{(0,1)}[x,x]-n\, a^{(0,1)}[x,x]^2-2\, n\, a^{(0,1)}[x,x]\, a^{(1,0)}[x,x]+\\
&\qquad (n\, a[x,x]+n0\, a0[x])\, a^{(1,1)}[x,x]\big)\big)\big)\Big/(1+n\, a[x,x]\, h[x]+n0\, a0[x]\, h[x])^3\Bigg)
\end{aligned}
$$

Functions ϕ's appearing in third derivatives

Functions $\phi_{3,1}$, $\phi_{3,2}$, and $\phi_{3,3}$ appearing in third pure derivatives

```
tmp = Simplify[
   D[g2[n1, n2, x1, x2, xp], {x1, 3}, {x2, 0}] /. {x1 → x, x2 → x} /. {n2 → n - n1}];
```

```
ϕ3,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1]]
```

$$
\begin{aligned}
&\big(a[x,xp]\\
&\quad\big(-12\, n\, n0\, a[x,x]\, a0'[x]\, h'[x]^2-12\, n0^2\, a0[x]\, a0'[x]\, h'[x]^2+6\, n^3\, a[x,x]^3\, h'[x]^3+\\
&\qquad 18\, n^2\, n0\, a[x,x]^2\, a0[x]\, h'[x]^3+18\, n\, n0^2\, a[x,x]\, a0[x]^2\, h'[x]^3+6\, n0^3\, a0[x]^3\, h'[x]^3+\\
&\qquad 3\, n0\, h'[x]\, a0''[x]+3\, n0\, a0'[x]\, h''[x]-6\, n^2\, a[x,x]^2\, h'[x]\, h''[x]-\\
&\qquad 12\, n\, n0\, a[x,x]\, a0[x]\, h'[x]\, h''[x]-6\, n0^2\, a0[x]^2\, h'[x]\, h''[x]+\\
&\qquad n\, a[x,x]\, h^{(3)}[x]+n0\, a0[x]\, h^{(3)}[x]-12\, n^2\, a[x,x]\, h'[x]^2\, a^{(1,0)}[x,x]-\\
&\qquad 12\, n\, n0\, a0[x]\, h'[x]^2\, a^{(1,0)}[x,x]+3\, n\, h''[x]\, a^{(1,0)}[x,x]+3\, n\, h'[x]\, a^{(2,0)}[x,x]+\\
&\qquad h[x]\,\big(-12\, n0^2\, a0'[x]^2\, h'[x]-6\, n^3\, a[x,x]^3\, h'[x]\, h''[x]-6\, n0^3\, a0[x]^3\, h'[x]\, h''[x]+\\
&\qquad\quad n0\, a0^{(3)}[x]+2\, n0^2\, a0[x]^2\, h^{(3)}[x]+n\, a^{(0,3)}[x,x]+6\, n\, n0^2\, a0[x]^2\, h'[x]^2\\
&\qquad\quad a^{(1,0)}[x,x]-12\, n^2\, h'[x]\, a^{(1,0)}[x,x]^2+6\, n0\, a0'[x]\, h'[x]\,\big(n^2\, a[x,x]^2\, h'[x]+\\
&\qquad\qquad 2\, n\, n0\, a[x,x]\, a0[x]\, h'[x]+n0^2\, a0[x]^2\, h'[x]-4\, n\, a^{(1,0)}[x,x]\big)+\\
&\qquad\quad 2\, n^2\, a[x,x]^2\,\big(-9\, n0\, a0[x]\, h'[x]\, h''[x]+h^{(3)}[x]+3\, n\, h'[x]^2\, a^{(1,0)}[x,x]\big)+\\
&\qquad\quad 2\, n\, n0\, a[x,x]\, a0[x]\\
&\qquad\qquad\big(-9\, n0\, a0[x]\, h'[x]\, h''[x]+2\, h^{(3)}[x]+6\, n\, h'[x]^2\, a^{(1,0)}[x,x]\big)+n\, a^{(3,0)}[x,x]\big)+\\
&\qquad h[x]^3\,\big(6\, n0^3\, a0'[x]^3+n0^3\, a0[x]^2\, a0^{(3)}[x]+n\, n0^2\, a0[x]^2\, a^{(0,3)}[x,x]+\\
&\qquad\quad 18\, n\, n0^2\, a0'[x]^2\, a^{(1,0)}[x,x]-6\, n\, n0^2\, a0[x]\, a0''[x]\, a^{(1,0)}[x,x]+\\
&\qquad\quad 6\, n^3\, a^{(1,0)}[x,x]^3-6\, n^2\, n0\, a0[x]\, a^{(1,0)}[x,x]\, a^{(2,0)}[x,x]-\\
&\qquad\quad 6\, n0\, a0'[x]\,\big(-3\, n^2\, a^{(1,0)}[x,x]^2+n\, a[x,x]\,\big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)+\\
&\qquad\qquad n0\, a0[x]\,\big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)\big)+n\, n0^2\, a0[x]^2\, a^{(3,0)}[x,x]+\\
&\qquad\quad n^2\, a[x,x]^2\,\big(n0\, a0^{(3)}[x]+n\,\big(a^{(0,3)}[x,x]+a^{(3,0)}[x,x]\big)\big)+\\
&\qquad\quad 2\, n\, a[x,x]\,\big(-3\, n\, a^{(1,0)}[x,x]\,\big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)+\\
&\qquad\qquad n0\, a0[x]\,\big(n0\, a0^{(3)}[x]+n\,\big(a^{(0,3)}[x,x]+a^{(3,0)}[x,x]\big)\big)\big)\big)+\\
&\qquad h[x]^2\,\big(n^3\, a[x,x]^3\, h^{(3)}[x]+n0^3\, a0[x]^3\, h^{(3)}[x]-6\,\big(n0\, a0'[x]+n\, a^{(1,0)}[x,x]\big)\\
&\qquad\quad \big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)-3\, n^2\, a[x,x]^2\,\big(n0\, a0'[x]\, h''[x]-n0\, a0[x]\, h^{(3)}[x]+\\
&\qquad\qquad n\, h''[x]\, a^{(1,0)}[x,x]+h'[x]\,\big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)\big)-3\, n0^2\, a0[x]^2\\
&\qquad\quad \big(h''[x]\,\big(n0\, a0'[x]+n\, a^{(1,0)}[x,x]\big)+h'[x]\,\big(n0\, a0''[x]+n\, a^{(2,0)}[x,x]\big)\big)+\\
&\qquad\quad 2\, n0\, a0[x]\,\big(3\, n0^2\, a0'[x]^2\, h'[x]+n0\, a0^{(3)}[x]+6\, n\, n0\, a0'[x]\, h'[x]\, a^{(1,0)}[x,x]+\\
&\qquad\qquad n\,\big(a^{(0,3)}[x,x]+3\, n\, h'[x]\, a^{(1,0)}[x,x]^2+a^{(3,0)}[x,x]\big)\big)+
\end{aligned}
$$

```
        n a[x, x] (6 n0^2 a0'[x]^2 h'[x] + 3 n0^2 a0[x]^2 h^(3)[x] -
            6 n0 a0'[x] (n0 a0[x] h''[x] - 2 n h'[x] a^(1,0)[x, x]) -
            6 n0 a0[x] (n h''[x] a^(1,0)[x, x] + h'[x] (n0 a0''[x] + n a^(2,0)[x, x])) +
            2 (n0 a0^(3)[x] + n (a^(0,3)[x, x] + 3 n h'[x] a^(1,0)[x, x]^2 + a^(3,0)[x, x]))))) +
    ((1 + n a[x, x] h[x] + n0 a0[x] h[x]) (e (1 + n a[x, x] h[x] + n0 a0[x] h[x])^3
        a^(0,3)[xp, x] + (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2
        (6 n0 a0'[x] h'[x] a^(1,0)[x, xp] - 6 n^2 a[x, x]^2 h'[x]^2 a^(1,0)[x, xp] -
          12 n n0 a[x, x] a0[x] h'[x]^2 a^(1,0)[x, xp] - 6 n0^2 a0[x]^2 h'[x]^2 a^(1,0)[x, xp] +
          3 n a[x, x] h''[x] a^(1,0)[x, xp] + 3 n0 a0[x] h''[x] a^(1,0)[x, xp] +
          6 n h'[x] a^(1,0)[x, x] a^(1,0)[x, xp] + 3 n a[x, x] h'[x] a^(2,0)[x, xp] +
          3 n0 a0[x] h'[x] a^(2,0)[x, xp] - a^(3,0)[x, xp] +
          h[x] (3 n^2 a[x, x]^2 (h''[x] a^(1,0)[x, xp] + h'[x] a^(2,0)[x, xp]) + 3 n0^2 a0[x]^2
               (h''[x] a^(1,0)[x, xp] + h'[x] a^(2,0)[x, xp]) + 3 (n0 a0''[x] a^(1,0)[x, xp] +
                 n a^(1,0)[x, xp] a^(2,0)[x, x] + (n0 a0'[x] + n a^(1,0)[x, x])
                  a^(2,0)[x, xp]) - 2 n0 a0[x] (3 n0 a0'[x] h'[x] a^(1,0)[x, xp] +
                 3 n h'[x] a^(1,0)[x, x] a^(1,0)[x, xp] + a^(3,0)[x, xp]) - 2 n a[x, x] (3 n0
                  a0'[x] h'[x] a^(1,0)[x, xp] + 3 n h'[x] a^(1,0)[x, x] a^(1,0)[x, xp] - 3 n0
                  a0[x] (h''[x] a^(1,0)[x, xp] + h'[x] a^(2,0)[x, xp]) + a^(3,0)[x, xp])) +
          h[x]^2 (-6 n0^2 a0'[x]^2 a^(1,0)[x, xp] + 3 n0^2 a0[x] a0''[x] a^(1,0)[x, xp] -
              6 n^2 a^(1,0)[x, x]^2 a^(1,0)[x, xp] + 3 n n0 a0[x] a^(1,0)[x, xp] a^(2,0)[x, x] +
              3 n n0 a0[x] a^(1,0)[x, x] a^(2,0)[x, xp] + 3 n0 a0'[x]
               (-4 n a^(1,0)[x, x] a^(1,0)[x, xp] + (n a[x, x] + n0 a0[x]) a^(2,0)[x, xp]) -
              n^2 a[x, x]^2 a^(3,0)[x, xp] - n0^2 a0[x]^2 a^(3,0)[x, xp] +
              n a[x, x] (3 n0 a0''[x] a^(1,0)[x, xp] + 3 n a^(1,0)[x, xp] a^(2,0)[x, x] +
                 3 n a^(1,0)[x, x] a^(2,0)[x, xp] - 2 n0 a0[x] a^(3,0)[x, xp]))))) /
     (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2) / (1 +
    n
     a[x, x]
     h[x] + n0
     a0[x]
     h[x])^4
```

```
ϕ3,2[n_, x_, xp_] = Simplify[Coefficient[tmp, n1^2]]
```

```
3 ((2 e n0 a0[xp] h[xp]^2 a^(0,1)[xp, x] a^(0,2)[xp, x]) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^4 +
    (e n^2 a[xp, x]^2 h[xp]^3 a^(0,1)[xp, x] a^(0,2)[xp, x]) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^4 +
    (2 e n0^2 a0[xp]^2 h[xp]^3 a^(0,1)[xp, x] a^(0,2)[xp, x]) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^4 -
    (e h[xp] a^(0,1)[xp, x] a^(0,2)[xp, x]) / (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^3 -
    (e n0 a0[xp] h[xp]^2 a^(0,1)[xp, x] a^(0,2)[xp, x]) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^3 -
    (e h[xp] a^(0,1)[xp, x] a^(0,2)[xp, x]) / (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^2 +
    (e n a[xp, x] h[xp]^2 (1 + 3 n0 a0[xp] h[xp]) a^(0,1)[xp, x] a^(0,2)[xp, x]) /
      (1 + n a[xp, x] h[xp] + n0 a0[xp] h[xp])^4 -
    (4 n0 h[x]^2 a0'[x] a^(0,1)[x, x] a^(1,0)[x, xp]) / (1 + n a[x, x] h[x] + n0 a0[x] h[x])^4 -
    (4 n n0 a[x, x] h[x]^3 a0'[x] a^(0,1)[x, x] a^(1,0)[x, xp]) /
```

$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(4\,n0^2\,a0[x]\,h[x]^3\,a0'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+\frac{2\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}-$$
$$\left(2\,n^2\,a[x,x]^2\,h[x]^2\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(4\,n\,n0\,a[x,x]\,a0[x]\,h[x]^2\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(2\,n0^2\,a0[x]^2\,h[x]^2\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+\frac{h[x]\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}+$$
$$\left(2\,n\,a[x,x]\,h[x]^2\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n0\,a0[x]\,h[x]^2\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(n^2\,a[x,x]^2\,h[x]^3\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n\,n0\,a[x,x]\,a0[x]\,h[x]^3\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(n0^2\,a0[x]^2\,h[x]^3\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(4\,n\,h[x]^2\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(4\,n^2\,a[x,x]\,h[x]^3\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4-$$
$$\left(4\,n\,n0\,a0[x]\,h[x]^3\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]\,a^{(1,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+\frac{2\,h[x]\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}+$$
$$\left(4\,n\,a[x,x]\,h[x]^2\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(4\,n0\,a0[x]\,h[x]^2\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n^2\,a[x,x]^2\,h[x]^3\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(4\,n\,n0\,a[x,x]\,a0[x]\,h[x]^3\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n0^2\,a0[x]^2\,h[x]^3\,a^{(1,0)}[x,xp]\,a^{(1,1)}[x,x]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\frac{h[x]\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}+$$
$$\left(2\,n\,a[x,x]\,h[x]^2\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n0\,a0[x]\,h[x]^2\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(n^2\,a[x,x]^2\,h[x]^3\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(2\,n\,n0\,a[x,x]\,a0[x]\,h[x]^3\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]\right)\Big/$$
$$(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\left(n0^2\,a0[x]^2\,h[x]^3\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,xp]\right)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4+$$
$$\Big(a[x,xp]\,\Big(-4\,n\,a[x,x]\,h'[x]^2\,a^{(0,1)}[x,x]-4\,n0\,a0[x]\,h'[x]^2\,a^{(0,1)}[x,x]+$$
$$h''[x]\,a^{(0,1)}[x,x]+h'[x]\,a^{(0,2)}[x,x]+2\,h'[x]\,a^{(1,1)}[x,x]+$$
$$h[x]\,\Big(-8\,n0\,a0'[x]\,h'[x]\,a^{(0,1)}[x,x]+2\,n^2\,a[x,x]^2\,h'[x]^2\,a^{(0,1)}[x,x]+$$
$$4\,n\,n0\,a[x,x]\,a0[x]\,h'[x]^2\,a^{(0,1)}[x,x]+2\,n0^2\,a0[x]^2\,h'[x]^2\,a^{(0,1)}[x,x]-$$
$$8\,n\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]+a^{(1,2)}[x,x]+a^{(2,1)}[x,x]\Big)-$$
$$h[x]^2\,\Big(n^2\,a[x,x]^2\,\Big(h''[x]\,a^{(0,1)}[x,x]+h'[x]\,\Big(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\Big)\Big)+$$
$$n0^2\,a0[x]^2\,\Big(h''[x]\,a^{(0,1)}[x,x]+h'[x]\,\Big(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\Big)\Big)+$$
$$2\,\Big(n0\,a0''[x]\,a^{(0,1)}[x,x]+n0\,a0'[x]\,\Big(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\Big)+$$

$$
\begin{aligned}
&n\left(a^{(1,0)}[x,x]\left(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\right)+\right.\\
&\quad\left.\left.\left.a^{(0,1)}[x,x]\left(a^{(0,2)}[x,x]+a^{(2,0)}[x,x]\right)\right)\right)\right)-2\,n0\,a0[x]\\
&\left(2\,n0\,a0'[x]\,h'[x]\,a^{(0,1)}[x,x]+2\,n\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]+\right.\\
&\quad\left.a^{(1,2)}[x,x]+a^{(2,1)}[x,x]\right)-2\,n\,a[x,x]\left(2\,n0\,a0'[x]\,h'[x]\,a^{(0,1)}[x,x]+\right.\\
&\quad 2\,n\,h'[x]\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]-n0\,a0[x]\left(h''[x]\,a^{(0,1)}[x,x]+\right.\\
&\quad\left.\left.\left.h'[x]\left(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\right)\right)+a^{(1,2)}[x,x]+a^{(2,1)}[x,x]\right)\right)+\\
&h[x]^3\left(6\,n0^2\,a0'[x]^2\,a^{(0,1)}[x,x]-2\,n0^2\,a0[x]\,a0''[x]\,a^{(0,1)}[x,x]-\right.\\
&\quad 2\,n\,n0\,a0[x]\,a^{(0,1)}[x,x]\,a^{(0,2)}[x,x]-2\,n\,n0\,a0[x]\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,x]+\\
&\quad 6\,n^2\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]^2-4\,n\,n0\,a0[x]\,a^{(1,0)}[x,x]\,a^{(1,1)}[x,x]-\\
&\quad 2\,n0\,a0'[x]\left(-6\,n\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]+n\,a[x,x]\left(a^{(0,2)}[x,x]+\right.\right.\\
&\quad\left.\left.2\,a^{(1,1)}[x,x]\right)+n0\,a0[x]\left(a^{(0,2)}[x,x]+2\,a^{(1,1)}[x,x]\right)\right)+n0^2\,a0[x]^2\\
&\quad a^{(1,2)}[x,x]-2\,n\,n0\,a0[x]\,a^{(0,1)}[x,x]\,a^{(2,0)}[x,x]+n0^2\,a0[x]^2\,a^{(2,1)}[x,x]+\\
&\quad n^2\,a[x,x]^2\left(a^{(1,2)}[x,x]+a^{(2,1)}[x,x]\right)-2\,n\,a[x,x]\left(n0\,a0''[x]\,a^{(0,1)}[x,x]+\right.\\
&\quad n\,a^{(0,2)}[x,x]\,a^{(1,0)}[x,x]+2\,n\,a^{(1,0)}[x,x]\,a^{(1,1)}[x,x]-\\
&\quad n0\,a0[x]\,a^{(1,2)}[x,x]+n\,a^{(0,1)}[x,x]\left(a^{(0,2)}[x,x]+a^{(2,0)}[x,x]\right)-\\
&\quad\left.\left.\left.\left.\left. n0\,a0[x]\,a^{(2,1)}[x,x]\right)\right)\right)\right)\Big/\left(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x]\right)^4\right)
\end{aligned}
$$

```
ϕ3,3[n_, x_, xp_] = Simplify[Coefficient[tmp, n1^3]]
```

$$
\begin{aligned}
&6\left(-\frac{e\,n\,a[xp,x]\,h[xp]^3\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^4}-\frac{e\,n0\,a0[xp]\,h[xp]^3\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^4}+\right.\\
&\quad\frac{e\,h[xp]^2\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^3}-\frac{h[x]^2\,a^{(0,1)}[x,x]^2\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}-\\
&\quad\frac{n\,a[x,x]\,h[x]^3\,a^{(0,1)}[x,x]^2\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}-\frac{n0\,a0[x]\,h[x]^3\,a^{(0,1)}[x,x]^2\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4}+\\
&\quad\left(a[x,xp]\,h[x]\,a^{(0,1)}[x,x]\left(-2\,h'[x]\,a^{(0,1)}[x,x]+\right.\right.\\
&\qquad h[x]\left(n\,a[x,x]\,h'[x]\,a^{(0,1)}[x,x]+n0\,a0[x]\,h'[x]\,a^{(0,1)}[x,x]-2\,a^{(1,1)}[x,x]\right)+\\
&\qquad h[x]^2\left(3\,n0\,a0'[x]\,a^{(0,1)}[x,x]+n\,a^{(0,1)}[x,x]^2+\right.\\
&\qquad\left.\left.\left.3\,n\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]-2\,(n\,a[x,x]+n0\,a0[x])\,a^{(1,1)}[x,x]\right)\right)\right)\Big/\\
&\quad\left.(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4\right)
\end{aligned}
$$

Functions $\phi_{2,1,1,1}$ and $\phi_{2,1,2,1}$ appearing in third mixed derivatives w.r.t. two strategies

```
tmp = Simplify[D[g3[n1, n2, n3, x1, x2, x3, xp], {x1, 2}, {x2, 1}] /.
    {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
```

```
ϕ2,1,1,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]]
```

$$
\begin{aligned}
&-\frac{2\, n\, a[x, xp]\, h[x]^2\, a^{(0,1)}[x, x]\, a^{(0,2)}[x, x]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^3} + \\
&\frac{2\, e\, (n\, a[xp, x] + n0\, a0[xp])\, h[xp]^2\, a^{(0,1)}[xp, x]\, a^{(0,2)}[xp, x]}{(1 + n\, a[xp, x]\, h[xp] + n0\, a0[xp]\, h[xp])^3} - \\
&\frac{2\, e\, h[xp]\, a^{(0,1)}[xp, x]\, a^{(0,2)}[xp, x]}{(1 + n\, a[xp, x]\, h[xp] + n0\, a0[xp]\, h[xp])^2} + \\
&\frac{h[x]\, a^{(0,2)}[x, x]\, a^{(1,0)}[x, xp]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^2} + \Big(a[x, xp]\, \big(h'[x]\, a^{(0,2)}[x, x] + \\
&\quad h[x]\, \big(-n\, a[x, x]\, h'[x]\, a^{(0,2)}[x, x] - n0\, a0[x]\, h'[x]\, a^{(0,2)}[x, x] + a^{(1,2)}[x, x]\big) + \\
&\quad h[x]^2\, \big(-2\, n0\, a0'[x]\, a^{(0,2)}[x, x] - 2\, n\, a^{(0,2)}[x, x]\, a^{(1,0)}[x, x] + \\
&\qquad (n\, a[x, x] + n0\, a0[x])\, a^{(1,2)}[x, x]\big)\big)\Big) \Big/ (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^3 - \\
&\Big(4\, h[x]\, (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])\, a^{(0,1)}[x, x] \\
&\quad \big((n\, a[x, x] + n0\, a0[x])\, h'[x] + h[x]\, \big(n0\, a0'[x] + n\, a^{(1,0)}[x, x]\big)\big)\, a^{(1,0)}[x, xp] - \\
&\quad 2\, (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^2\, a^{(1,0)}[x, xp] \\
&\quad \big(h'[x]\, a^{(0,1)}[x, x] + h[x]\, a^{(1,1)}[x, x]\big) - h[x]\, (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^2 \\
&\quad a^{(0,1)}[x, x]\, a^{(2,0)}[x, xp] + a[x, xp]\, \big(-6\, h[x]\, a^{(0,1)}[x, x] \\
&\qquad \big((n\, a[x, x] + n0\, a0[x])\, h'[x] + h[x]\, \big(n0\, a0'[x] + n\, a^{(1,0)}[x, x]\big)\big)^2 + \\
&\qquad 4\, (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])\, \big((n\, a[x, x] + n0\, a0[x])\, h'[x] + \\
&\qquad\quad h[x]\, \big(n0\, a0'[x] + n\, a^{(1,0)}[x, x]\big)\big)\, \big(h'[x]\, a^{(0,1)}[x, x] + h[x]\, a^{(1,1)}[x, x]\big) + \\
&\qquad 2\, h[x]\, (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])\, a^{(0,1)}[x, x] \\
&\qquad \big(2\, n0\, a0'[x]\, h'[x] + n\, a[x, x]\, h''[x] + n0\, a0[x]\, h''[x] + 2\, n\, h'[x]\, a^{(1,0)}[x, x] + \\
&\qquad\quad h[x]\, \big(n0\, a0''[x] + n\, a^{(2,0)}[x, x]\big)\big) - (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^2 \\
&\qquad \big(h''[x]\, a^{(0,1)}[x, x] + 2\, h'[x]\, a^{(1,1)}[x, x] + h[x]\, a^{(2,1)}[x, x]\big)\big)\Big) \Big/ \\
&\quad (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^4
\end{aligned}
$$

```
ϕ2,1,2,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1^2 n2]]
```

$$
\begin{aligned}
&-\frac{6\, e\, n\, a[xp, x]\, h[xp]^3\, a^{(0,1)}[xp, x]^3}{(1 + n\, a[xp, x]\, h[xp] + n0\, a0[xp]\, h[xp])^4} - \\
&\frac{6\, e\, n0\, a0[xp]\, h[xp]^3\, a^{(0,1)}[xp, x]^3}{(1 + n\, a[xp, x]\, h[xp] + n0\, a0[xp]\, h[xp])^4} + \frac{6\, e\, h[xp]^2\, a^{(0,1)}[xp, x]^3}{(1 + n\, a[xp, x]\, h[xp] + n0\, a0[xp]\, h[xp])^3} - \\
&\frac{4\, h[x]^2\, a^{(0,1)}[x, x]^2\, a^{(1,0)}[x, xp]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^4} - \frac{4\, n\, a[x, x]\, h[x]^3\, a^{(0,1)}[x, x]^2\, a^{(1,0)}[x, xp]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^4} - \\
&\frac{4\, n0\, a0[x]\, h[x]^3\, a^{(0,1)}[x, x]^2\, a^{(1,0)}[x, xp]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^4} - \frac{2\, h[x]^2\, a^{(0,1)}[x, x]^2\, a^{(1,0)}[x, xp]}{(1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^3} + \\
&\Big(6\, a[x, xp]\, h[x]\, a^{(0,1)}[x, x]\, \big(-2\, h'[x]\, a^{(0,1)}[x, x] + \\
&\quad h[x]\, \big(n\, a[x, x]\, h'[x]\, a^{(0,1)}[x, x] + n0\, a0[x]\, h'[x]\, a^{(0,1)}[x, x] - 2\, a^{(1,1)}[x, x]\big) + \\
&\quad h[x]^2\, \big(3\, n0\, a0'[x]\, a^{(0,1)}[x, x] + n\, a^{(0,1)}[x, x]^2 + 3\, n\, a^{(0,1)}[x, x]\, a^{(1,0)}[x, x] - \\
&\qquad 2\, (n\, a[x, x] + n0\, a0[x])\, a^{(1,1)}[x, x]\big)\big)\Big) \Big/ (1 + n\, a[x, x]\, h[x] + n0\, a0[x]\, h[x])^4
\end{aligned}
$$

Function $\phi_{1,1,1,1,1,1}$ appearing in third mixed derivatives w.r.t. three strategies

```
tmp = Simplify[D[g4[n1, n2, n3, n4, x1, x2, x3, x4, xp], {x1, 1}, {x2, 1}, {x3, 1}] /.
    {x1 → x, x2 → x, x3 → x, x4 → x} /. {n4 → n - n1 - n2 - n3}];
```

```
ϕ1,1,1,1,1,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2 n3]]
```

$$6\left(-\frac{e\,n\,a[xp,x]\,h[xp]^3\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^4}-\frac{e\,n0\,a0[xp]\,h[xp]^3\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^4}+\frac{e\,h[xp]^2\,a^{(0,1)}[xp,x]^3}{(1+n\,a[xp,x]\,h[xp]+n0\,a0[xp]\,h[xp])^3}-\frac{h[x]^2\,a^{(0,1)}[x,x]^2\,a^{(1,0)}[x,xp]}{(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^3}+\Big(a[x,xp]\,h[x]\,a^{(0,1)}[x,x]\,\big(-2\,h'[x]\,a^{(0,1)}[x,x]+h[x]\,(n\,a[x,x]\,h'[x]\,a^{(0,1)}[x,x]+n0\,a0[x]\,h'[x]\,a^{(0,1)}[x,x]-2\,a^{(1,1)}[x,x])+h[x]^2\,(3\,n0\,a0'[x]\,a^{(0,1)}[x,x]+n\,a^{(0,1)}[x,x]^2+3\,n\,a^{(0,1)}[x,x]\,a^{(1,0)}[x,x]-2\,(n\,a[x,x]+n0\,a0[x])\,a^{(1,1)}[x,x])\big)\Big)\Big/(1+n\,a[x,x]\,h[x]+n0\,a0[x]\,h[x])^4\right)$$

Symmetries due to property P3

$\phi_{1,2,1,1} = \phi_{2,1,1,1}$

```
tmp = Simplify[D[g3[n1, n2, n3, x1, x2, x3, xp], {x1, 1}, {x2, 2}] /.
    {x1 → x, x2 → x, x3 → x} /. {n3 → n - n1 - n2}];
ϕ1,2,1,1[n_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2]];
```

```
Simplify[ϕ1,2,1,1[n, x, xp] - ϕ2,1,1,1[n, x, xp]]
```

0

$\phi_{1,2,1,2} = \phi_{2,1,2,1}$

```
ϕ1,2,1,2[n_, x_, xp_] = Simplify[Coefficient[tmp, n1 n2^2]];
Simplify[ϕ1,2,1,2[n, x, xp] - ϕ2,1,2,1[n, x, xp]]
```

0

Links between k- and k'-index-ϕ's, k'<k

$\phi_{1,1,1,1} = \phi_{2,2}$

```
Simplify[ϕ1,1,1,1[n, x, xp] - ϕ2,2[n, x, xp]]
```

0

$\phi_{2,1,1,1} = \phi_{3,1} / 3$

```
Simplify[ϕ2,1,1,1[n, x, xp] - ϕ3,2[n, x, xp] / 3]
```

0

$\phi_{2,1,2,1} = \phi_{3,3}$

```
Simplify[ϕ2,1,2,1[n, x, xp] - ϕ3,3[n, x, xp]]
```

0

$\phi_{1,1,1,1,1,1} = \phi_{3,3}$

```
Simplify[ϕ1,1,1,1,1,1[n, x, xp] - ϕ3,3[n, x, xp]]
```

0

Consistency with the monomorphic g-function

```
Simplify[D[g1[n, x, xp], {x, 1}] - ϕ1,1[n, x, xp] n]
```

0

```
Simplify[D[g1[n, x, xp], {x, 2}] - (ϕ2,1[n, x, xp] n + ϕ2,2[n, x, xp] n^2)]
```

0

```
Simplify[
 D[g1[n, x, xp], {x, 3}] - (ϕ3,1[n, x, xp] n + ϕ3,2[n, x, xp] n^2 + ϕ3,3[n, x, xp] n^3)]
```

0

Other examples of property P4

```
Simplify[
 (D[g2[n1, n2, x1, x2, xp], {x2, 1}] /. {x1 → x, x2 → x}) - ϕ1,1[n1 + n2, x, xp] n2]
```

0

```
Simplify[(D[g2[n1, n2, x1, x2, xp], {x1, 1}, {x2, 1}] /. {x1 → x, x2 → x}) -
  ϕ1,1,1,1[n1 + n2, x, xp] n1 n2]
```

0

```
Simplify[(D[g2[n1, n2, x1, x2, xp], {x1, 1}, {x2, 2}] /. {x1 → x, x2 → x}) -
  (ϕ1,2,1,1[n1 + n2, x, xp] n1 n2 + ϕ1,2,1,2[n1 + n2, x, xp] n1 n2^2)]
```

0

```
Simplify[(D[g3[n1, n2, n3, x1, x2, x3, xp], {x2, 1}] /. {x1 → x, x2 → x, x3 → x}) -
  ϕ1,1[n1 + n2 + n3, x, xp] n2]
```

0

```
Simplify[(D[g4[n1, n2, n3, n4, x1, x2, x3, x4, xp], {x2, 1}, {x3, 1}] /.
    {x1 → x, x2 → x, x3 → x, x4 → x}) - ϕ1,1,1,1[n1 + n2 + n3 + n4, x, xp] n2 n3]
```

0

Clear symbols

```
Clear[g1, g2, g3, g4, tmp]
Unset[{ϕ1,1[n_, x_, xp_], ϕ2,1[n_, x_, xp_], ϕ2,2[n_, x_, xp_],
   ϕ1,1,1,1[n_, x_, xp_], ϕ3,1[n_, x_, xp_], ϕ3,2[n_, x_, xp_],
   ϕ3,3[n_, x_, xp_], ϕ2,1,1,1[n_, x_, xp_], ϕ2,1,2,1[n_, x_, xp_],
   ϕ1,1,1,1,1,1[n_, x_, xp_], ϕ1,2,1,1[n_, x_, xp_], ϕ1,2,1,2[n_, x_, xp_]}];
```